Unipolar Multiplication in Stochastic Computing

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***Abstract*—In this article, we will discuss the relationship between properties of a bitstream and the associated error. Namely, we will show the correspondence between the length the and associated error of the bitstream as well as the fractional value and the associated error. Error decreases exponentially as the length of the sequences increases. Error has a parabolic relationship with error when two sequences are multiplied together where bitsreams closer to $\frac{1}{2}$ have highest absolute error.**

Keywords*—*stochastic computing

I. Introduction

Stochastic computing is a potential alternative to traditional computing due to its low power costs, computing efficiency, memory persistence, and simplicity. It introduces randomness and therefore excels in applications which allow for tolerable error and prefer high efficiency.

In this paper, we discuss the error that arises when using the unipolor multiplication of two bitstreams, which we define as stochastic sequences of a given length. In the first section, we explore the effects of increasing the bitstream length for different fractional values. In the second section, we examine the multiplication of two bitstreams of a set length but with different fractional values.

Values are represented in the stochastic domain by the probability of a '1' appearing in their sequence. For instance, to represent the fraction $\frac{1}{3}$ for a bitstream of length 9, exactly three 1's will appear. It is remedial to show that this representation is non-unique, which will be utilized throughout this paper. This encoding scheme allows for mathematical operations using logical gates. Thus, in order to perform unipolar multiplication of two sequences, the AND gate is used for the bitwise operation. Similarly, scaled addition and bipolar multiplication are obtained through the use of a MUX and a XNOR..

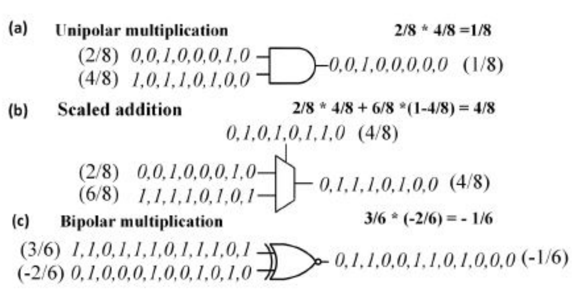

Figure 1.1: The figure above shows examples of mathematical operations for two encoded sequences. (a) shows unipolar multiplication is calculated using an AND gate, (b) shows scaled addition is calculated using a MUX, and (c) shows that bipolar multiplication is calculated using an XNOR.

This paper attempts to address the accuracy issues of SC by examining the case of unipolar multiplication. Since this operation is a bitwise AND operation, the operation will have a multinomial distribution. For simplicity, we will model the generation of encoded sequences as a random process that produces Bernoulli random variables with probability p. Error occurs when the result of the operation is different than then theoretical result. Absolute error is calculated by first squaring the difference and then taking the square root of the result.

II. Calculating Error

In order to calculate error between multiplying two sequences, all permutations of the sequences are multiplied. That is, every possible way to represent one fraction is operated with the bitwise AND operation with every possible way to represent the second. For each combination, the result is then subtracted from the expected result and recorded. Mathematically, the operation is as follows:

We first define a bitstream of length n to be:

$$X_n = (x_1, x_2, ..., x_n)$$

The theoretical value from multiplying two sequences X and Y is found by multiplying the fractional values of both sequences:

$$E_{theoretical} = \frac{\left(\sum_{i=1}^{i=n} x_i\right)\left(\sum_{i=1}^{i=n} y_i\right)}{n^2} \tag{1}$$

However, in stochastic computing, the two random sequences will very rarely yield the theoretical result. Instead, the two bitsreams X and Y with respective probabilities p_1 and p_2 will result in a range of values according to a multinomial distribution. Moreover, we assume that $p_1 > p_2$. Let U be the result of multiplying X and Y:

$$W = (x_1y_1, x_2y_2, ..., x_ny_n) = (w_1, w_2, ..., w_n)$$

And let $Z = XY$ be a random variable that multiplying these two bitstreams can take on:

$$Z = \frac{1}{n}\sum_{i=1}^{n} w_i \tag{2}$$

Therefore, the probability that $Z = XY$ is a given fraction z is defined by:

$$P(Z = z) = \frac{\binom{n}{np_1}\binom{np_1}{nz}\binom{n-np_1}{np_2-nz}}{\binom{n}{np_1}\binom{n}{np_2}} \tag{3}$$

where z has a range of values from $max(0, p_1 + p_2 - 1)$ to p_2.

Thus, the average absolute error is calculated by:

$$\mathbf{E}|\ Error\ | = \sum_{z=max(0,np_1+np_2-n)}^{np_2} P(z)\sqrt{(nE_{theoretical} - z)^2} \quad (4)$$

Following, the variance of the absolute error can be derrived:

$$\text{VAR}(|\ Error\ |) = \mathbf{E}[|\ Error\ |^2] - (\mathbf{E}[|\ Error\ |])^2 \quad (5)$$

To clarify the above equations, consider the example of two bitstreams X and Y with $n = 3$. Furthermore, let X and Y have respective probabilities $p_1 = \frac{1}{3}$ and $p_2 = \frac{2}{3}$. Therefore, the theoretical result $E_{theoretical} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{9}$. In order to calculate, the average error, first consider all permutations of each sequence:

$$X = (1,0,0), (0,1,0), (1,0,0)$$

$$Y = (0,1,1), (1,0,1), (1,1,0)$$

Then, each representation of X is operated with the bitwise AND operation with each representation of Y and the number of 1's are summed. The results of the 9 multiplications are then recorded:

$$[0,1,1,1,0,1,0,1,1]$$

The absolute error of each multiplication is calculated:

$$\left[\frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}\right]$$

The average and variance of these values can be found:

$$\mathbf{E}[|\ Error\ |] = \frac{1}{9}\left(\frac{4}{3}\right) = \frac{4}{27}$$

$$\text{VAR}(|\ Error\ |) = \sqrt{\frac{1}{9}\left(\frac{2}{81}\right)} = \sqrt{\frac{2}{729}}$$

III. Length and Unipolar Multiplication

The probability of error is dependent on the lengths of the sequences operated upon. Generally, as the length of the bitstreams increases, the probability of large error decreases. In this section, we use the method of calculating the mean and variation of the absolute error shown previously.

To show this trend, we will find the average and mean error for multiplying two sequences and then vary the lengths of these sequences. In the figures below, for each length of the sequence n, three different fractions are multiplied together: $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right), \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$, and $\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)$. The lengths, n are incremented from 4 to 128 with a step of 4 for the mean and variance.

Likewise, the normalized error, or the percent error, can be found by dividing each error by the theoretical value, $E_{theoretical}$. Figure 3.3 and Figure 3.4 show the normalized average error and standard deviation.

From the figures, it is concluded that as the length of the bitstream is increased, the mean and standard deviation of the absolute error decreases for the same fractions represented by the sequences. Moreover, the mean of the error decreases exponentially whereas the variation decreases rapidly and then remains relatively constant.

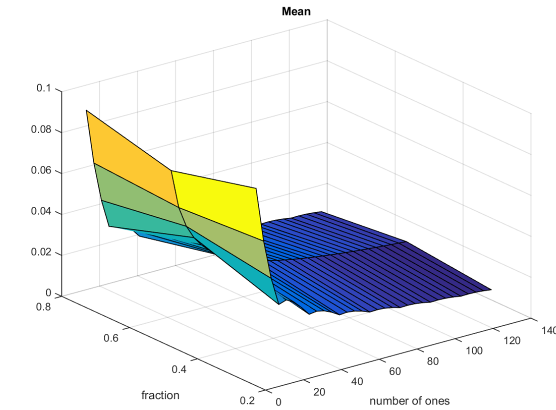

Figure 3.1: The figure above shows the mean of absolute error for $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2$, and $\left(\frac{3}{4}\right)^2$ for lengths $n = 4$ to $n = 128$ with a step of 4.

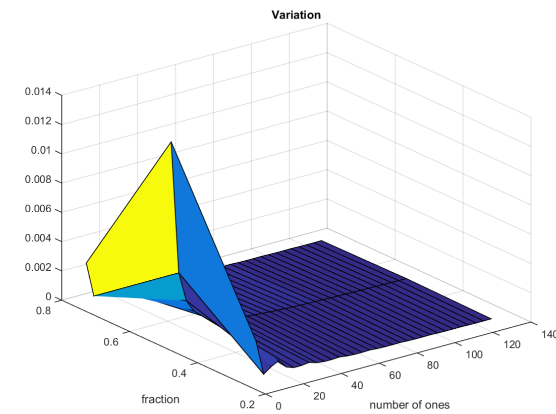

Figure 3.2: The figure above shows the variation of absolute error for $\left(\frac{1}{4}\right)^2, \left(\frac{1}{2}\right)^2$, and $\left(\frac{3}{4}\right)^2$ for lengths $n = 4$ to $n = 128$ with a step of 4.

The values found were confirmed by finding every representation of the sequence and multiplying it with one another. For instance, for a bitstream of length $n = 4$, the 6 representations of $\frac{1}{2}$ were multiplied with one another. Then, the results were compared with the theoretical value ($\frac{1}{4}$) to find the mean and standard deviation. Due to rapid increase in permutations for larger n, this brute force method was only able to confirm results for up to $n = 12$.

IV. Fractional Values and Unipolar Multiplication

Next, we find the mean and standard deviation of the absolute error by comparing different fractional values while maintaining constant sequence length. In other words, we will compare the effects of multiplying different fractions with one another and keep the length of the bitstream the same. The same methods to find mean and variation of multiplying two bitstreams are used. Generally, multiplying two fractions near $\frac{1}{2}$ will result in higher absolute error but lower normalized error.

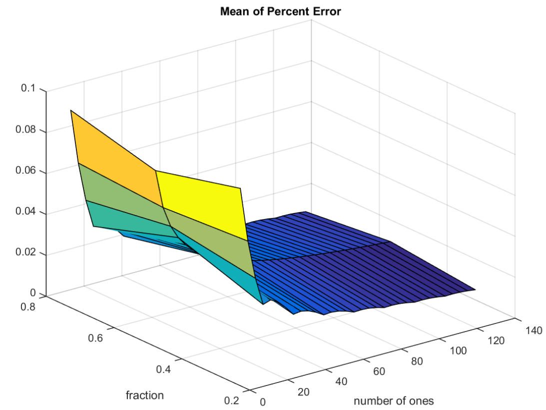

Figure 3.3: The figure above shows the mean of the normalized absolute error for $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, and $\left(\frac{3}{4}\right)^2$ for lengths $n = 4$ to $n = 128$ with a step of 4.

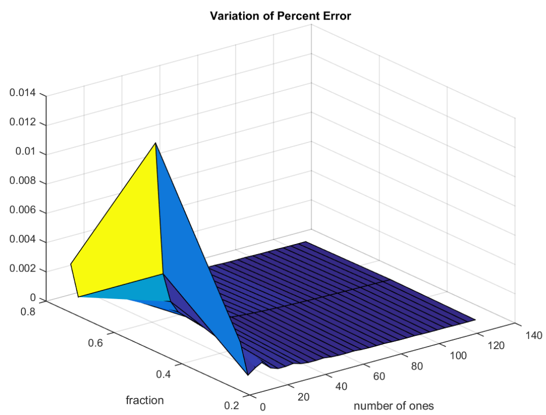

Figure 3.4: The figure above shows the variation of the normalized absolute error for $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, and $\left(\frac{3}{4}\right)^2$ for lengths $n = 4$ to $n = 128$ with a step of 4.

Figure 4.1 and Figure 4.2 shows the mean and variation of the absolute error for $n = 64$. Similarly, Figure 4.3 and Figure 4.4 show the normalized mean and standard deviation. Again, the relative error is found by dividing by the theoretical value.

It is evident from the figures that error is maximized as the multiplied fractions approach $\frac{1}{2}$ as both the mean and standard deviation of the absolute error is maximum. However, relative error has an opposite trend with minimum error near the center and higher relative error when smaller fractions are multiplied. This large error near $frac12$ is due to the nature of multinomials and the maximum number of permutations at this value.

As before, the results were confirmed with a brute force approach by producing every permutation of each fraction for a given n and finding the mean and standard deviation. In this case, up to $n = 12$ was used to confirm the results of the multinomial formula.

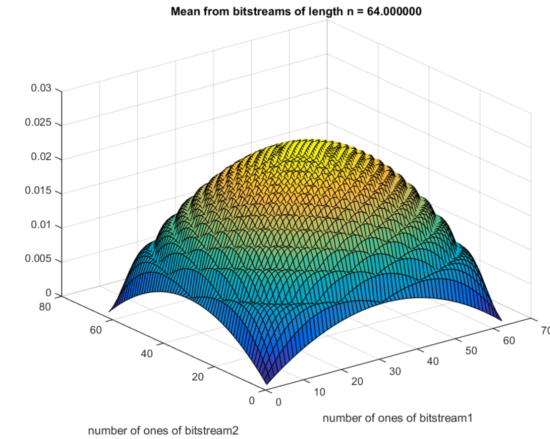

Figure 4.1: The figure above shows the mean of the absolute error for $n = 64$. Each fraction is multiplied with each other fraction to determine the absolute error.

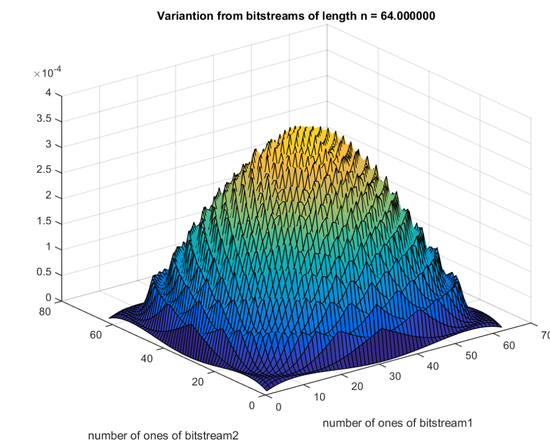

Figure 4.2: The figure above shows the variation of the absolute error for $n = 64$. Each fraction is multiplied with each other fraction to determine the absolute error.

V. Length and Fractions

In general, the longer the sequences, the lower chance of error. Moreover, multiplying fractions that are furthest from $\frac{1}{2}$ will result in lower chances of error.

The graphs in Figure 5.1 present the results for multiplying different fractions. On each plot, the different graphs represent different bitstream lengths ($n = 32, 64,$ and 512).

VI. Conclusion

Throughout this paper, we showed the effects of changing the length of the sequences and the fractions they represent. By both symbolic derivation and methodological simulation, we determined that the absolute error decreases exponentially as length increases and the error decreases parabolically as represented fractions tend away from $\frac{1}{2}$. Moreover, the percent error, which refers to the error when compared to the theoretical results, has the same relationship as absolute error for increasing sequence length but an inverse relationship for fractional representation. Thus, to have a high confidence level

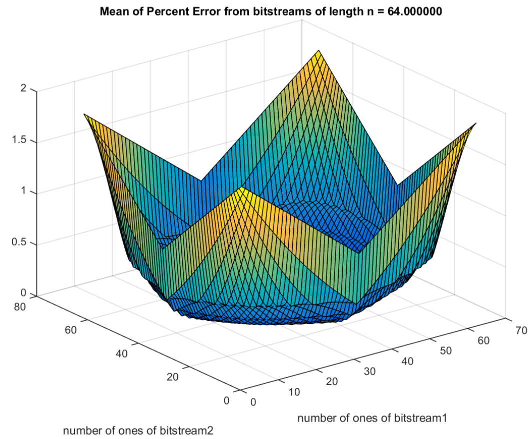

Figure 4.3: The figure above shows the mean of the normalized absolute error for $n = 64$. Each fraction is multiplied with each other fraction to determine the absolute and relative error.

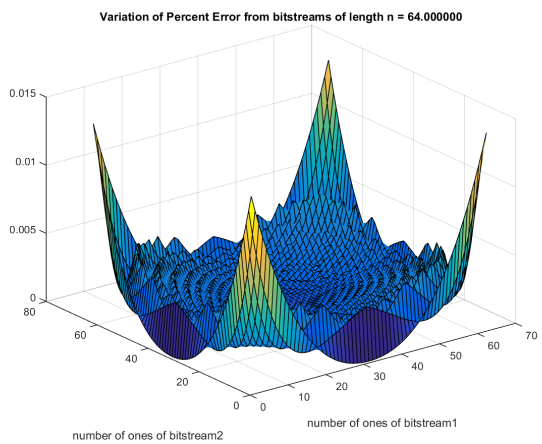

Figure 4.4: The figure above shows the variation of the normalized absolute error for $n = 64$. Each fraction is multiplied with each other fraction to determine the absolute and relative error.

when multiplying two stochastic sequences, we should use longer bitstreams and avoid fractions near $\frac{1}{2}$.

Acknowledgment

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References

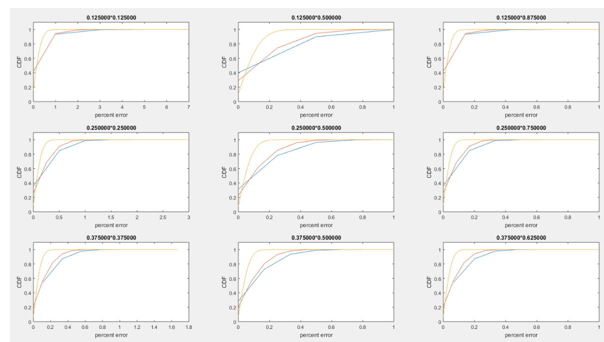

Figure 5.1: The above figure shows the different CDF's versus percent error for different multiplying fractions. Each plot shows the cumulative density function for error for $n = 32$ (blue), $n = 64$ (red), and $n = 512$ (yellow).