Lagrangian Relaxation

NanoCad Tutorial 4/17/2012

Lagrangian Relaxation

- Suppose that

$$\text{minimize } f(x), \text{ subject to } f_i(x) \leqslant 0$$

  Is difficult to solve

- However, suppose that

$$\text{minimize } f(x) + f_i(x)$$

  Is easy

- Then consider Lagrangian Relaxation

Lagrangian Relaxation

- Idea: Optimize the Lagrangian of

$$\text{minimize } f(x), \text{ subject to } f_i(x) \leqslant 0$$

Which is:

$$L(x,\lambda) = f(x) + \sum_i \lambda_i f_i(x)$$

- Finally, instead of solving:

$$\text{minimize } f(x), \text{ subject to } f_i(x) \leqslant 0$$

Solve

$$\max_{\lambda} \min_{x} (L(x,\lambda))$$

Lagrangian Relaxation Details

- Details

$$\max_{\lambda} \min_{x} \left(L\left(x, \lambda \right) \right)$$

  Has two optimizations (min x, max lambda)

  - This is only good if min x is very efficient

- When are

$$\max_{\lambda} \min_{x} \left(L\left(x, \lambda \right) \right)$$

$$\text{minimize } f\left(x \right), \text{subject to } f_{i}\left(x \right) \leqslant 0$$

  Equivalent?

  - Sufficient – convex and strictly feasible; Necessary: ?

Everett's Theorem

Everett's Theorem Let us start with a very elementary observation. Suppose we have maximized $L(\cdot, u)$ for a certain u, and thus obtained an optimal x_u, together with its constraint-value $c(x_u) \in \mathbb{R}^m$; set $g_u := -c(x_u)$ and take an arbitrary $x \in \mathcal{X}$ such that $c(x) = -g_u$. By definition, $L(x, u) \leqslant L(x_u, u)$, which can be written

$$f(x) \leqslant f(x_u) - u^\top c(x_u) + u^\top c(x) = f(x_u)\,. \tag{23}$$

This is a useful result for approximate resolutions of (1):

Theorem 12 (Everett) *With the notation above, x_u solves the following perturbation of* (1)*:*

$$\max f(x)\,, \quad x \in \mathcal{X}\,, \quad c(x) = -g_u\,. \qquad \square$$

Examples

- Integer Programming

$$\min \sum_{i,j=1}^{n} Q_{ij} x_i x_j = x^\top Q x \,, \quad x_i^2 = 1, \; i = 1, \ldots, n \,.$$

becomes

$$L(x,u) = x^\top Q x - \sum_{i=1}^{n} u_i (x_i^2 - 1) = x^\top (Q - D(u)) x + e^\top u \,;$$

which is just

$$\max e^\top u \,, \quad \text{subject to } Q - D(u) \succcurlyeq 0 \,.$$

Gate Sizing (Network Optimization Problem)

$$\text{minimize} \quad \sum_g p_g w_g$$

$$\text{Subject to} \quad t_{a(g)} + d_i(g, w) \leqslant t_{a(g')} \text{ for } g' \in \text{fanout(g)}$$

$$t_j \leqslant T_{\max}$$

$$L(w, t_a, \lambda) = \sum_{g \in \mathcal{G}} p_g w_g + \sum_g \sum_{g' \in \mathrm{fo}(g)} \lambda_{g,g'} (t_{a(g)} + d(g,w) - t_{a(g')}) + \sum_{g \in \mathrm{PO}} \lambda_g (t_{a(g)} - T_{\max}).$$

Gate Sizing

$$
\begin{aligned}
L(w, t_a, \lambda) = & \sum_{g \in \mathcal{G}} p_g w_g \\
& + \sum_{g} \sum_{g' \in \mathrm{fo}(g)} \lambda_{g,g'} (t_{a(g)} + d(g, w) - t_{a(g')}) \\
& + \sum_{g \in \mathrm{PO}} \lambda_g (t_{a(g)} - T_{\max}).
\end{aligned}
$$

$$
\begin{aligned}
L(w, t_a, \lambda) = & \sum_{g \in \mathcal{G}} \left(\sum_{g'' \in \mathrm{fo}(g)} \lambda_{g,g''} - \sum_{g' \in \mathrm{fi}(g)} \lambda_{g',g} \right) \cdot t_{a(g)} \\
& + \sum_{g \in \mathcal{G}} p_g w_g + \left(\sum_{g \in \mathrm{PO}} \lambda_g \right) \cdot T_{\max} \\
& + \sum_{g} \left(\sum_{g' \in \mathrm{fo}(g)} \lambda_{g,g'} \right) d(g, w)
\end{aligned}
$$

Gate Sizing

$$L(w, t_a, \lambda) = \sum_{g \in \mathcal{G}} p_g w_g + \sum_{g} \left(\sum_{g' \in \mathrm{fo}(g)} \lambda_{g,g'} \right) d(g, w) + \left(\sum_{g \in \mathrm{PO}} \lambda_g \right) \cdot T_{\max}.$$

$$\max_{\lambda \geq 0} \left\{ \min_{t_a, w_{\min} \leq w \leq w_{\max}} \{ L(w, t_a, \lambda) \} \right\}.$$

Gate Sizing

- Minimizing over w

$$L(w, t_a, \lambda) = \sum_{g \in \mathcal{G}} p_g w_g + \sum_{g} \left(\sum_{g' \in \mathrm{fo}(g)} \lambda_{g,g'} \right) d(g, w) + \left(\sum_{g \in \mathrm{PO}} \lambda_g \right) \cdot T_{\max}.$$

Is very fast. Method iterates between minimizing over w and maximizing over lambda

Summary

- Difficult problems may benefit from Lagrangian Relaxation
  - Problems where minimizing with constraint in objective is easy
- Useful in network flow problems, combinatorial problems
- Can also be used to generate theoretical bounds